Generalizations of Guillera-Sondow's double integral formulas

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Abstract. We evaluate certain multidimensional integrals in terms of the Lerch transcendent function Φ, generalizing Guillera-Sondow's formulas. As an application, we get new representations of classical constants like Euler's constant γ and $\ln(4/\pi)$.

The *Lerch transcendent* Φ is defined as the analytic continuation of the series

$$\Phi(z,s,u)=\frac{1}{u^s}+\frac{z}{(u+1)^s}+\frac{z^2}{(u+2)^s}+\cdots,$$

which converges for any complex number u with $\operatorname{Re} u>0$ if z and s are any complex numbers with either $|z|<1$, or $|z|=1$ and $\operatorname{Re} s>1$ (we suppose $\zeta^s=\exp(s\log\zeta)$, where $\log\zeta$ is the principal branch of the logarithm). The function Φ is holomorphic in z and s, for $z\in\mathbb{C}\backslash[1,\infty]$ and all complex s (see [1, section 1.11] or [2, section 2]). From the definition it follows that

$$\Phi(z,s,u+1)=\frac{1}{z}\left(\Phi(z,s,u)-\frac{1}{u^s}\right), \tag{1}$$

$$\Phi(z,s+1,u)=-\frac{1}{s}\frac{\partial\Phi}{\partial u}(z,s,u). \tag{2}$$

In the paper [2] J. Sondow and J. Guillera proved the following two theorems.

Theorem 1 *Suppose $u>0$, $v>0$, $u\neq v$, either $z\in\mathbb{C}\backslash[1,\infty)$ and $\operatorname{Re} s>-2$, or $z=1$ and $\operatorname{Re} s>-1$. Then*

$$\int_{[0,1]^2}\frac{x_1^{u-1}x_2^{v-1}}{1-zx_1x_2}(-\ln x_1x_2)^s dx_1dx_2=\Gamma(s+1)\frac{\Phi(z,s+1,v)-\Phi(z,s+1,u)}{u-v},$$

$$\int_{[0,1]^2}\frac{(x_1x_2)^{u-1}}{1-zx_1x_2}(-\ln x_1x_2)^s dx_1dx_2=\Gamma(s+2)\Phi(z,s+2,u).$$

Theorem 2 *Suppose $u>0$, either $z\in\mathbb{C}\backslash[1,\infty)$ and $\operatorname{Re} s>-3$, or $z=1$ and $\operatorname{Re} s>-2$. Then*

$$\int_{[0,1]^2}\frac{1-x_1}{1-zx_1x_2}(x_1x_2)^{u-1}(-\ln x_1x_2)^s dx_1dx_2$$
$$=\Gamma(s+2)\left[\Phi(z,s+2,u)+\frac{(1-z)\Phi(z,s+1,u)-u^{-s-1}}{z(s+1)}\right].$$

The purpose of this paper is to prove the following m-dimensional analogs of Theorems 1 and 2 (in what follows $d\overline{x}$ means $dx_1dx_2\cdots dx_m$, where m is the dimension of an integral).

Theorem 3 *Suppose m is a positive integer, $\operatorname{Re} u>0$, $\operatorname{Re} v>0$, $u\neq v$, either $z\in\mathbb{C}\backslash[1,\infty)$ and $\operatorname{Re} s>-m$, or $z=1$ and $\operatorname{Re} s>1-m$. For the case $m>1$ define the function*

$$F_{m,u,v}(x_1,x_2,\ldots,x_m)=(x_1x_2\cdots x_m)^{v-1}(x_1^{u-v}+(x_1x_2)^{u-v}+\cdots+(x_1x_2\cdots x_{m-1})^{u-v}).$$

Then

$$\int_{[0,1]^m}\frac{F_{m,u,v}(x_1,x_2,\ldots,x_m)}{1-zx_1x_2\cdots x_m}(-\ln x_1x_2\cdots x_m)^s d\overline{x}$$
$$=\frac{\Gamma(s+m-1)}{(m-2)!}\cdot\frac{\Phi(z,s+m-1,v)-\Phi(z,s+m-1,u)}{u-v}\quad \textit{for } m>1, \tag{3}$$

$$\int_{[0,1]^m}\frac{(x_1x_2\cdots x_m)^{u-1}}{1-zx_1x_2\cdots x_m}(-\ln x_1x_2\cdots s_m)^s d\overline{x}=\frac{\Gamma(s+m)}{(m-1)!}\Phi(z,s+m,u)\quad \textit{for } m\geq 1. \tag{4}$$

Theorem 4 *Suppose m is an integer >1, $\operatorname{Re} u>0$, either $z\in\mathbb{C}\backslash[1,\infty)$ and $\operatorname{Re} s>-m-1$, or $z=1$ and $\operatorname{Re} s>-m$. Then*

$$\int_{[0,1]^m}\frac{m-1-x_1-x_1x_2-\cdots-x_1x_2\cdots x_{m-1}}{1-zx_1x_2\cdots x_m}(x_1x_2\cdots x_m)^{u-1}(-\ln x_1x_2\cdots x_m)^s\,d\overline{x}$$
$$=\frac{\Gamma(s+m)}{(m-2)!}\left[\Phi(z,s+m,u)+\frac{(1-z)\Phi(z,s+m-1,u)-u^{-s-m+1}}{z(s+m-1)}\right]. \quad (5)$$

In the case $m=2$ Theorems 3 and 4 give Theorems 1 and 2. As an example, we give also the case $m=3$.

Example 1 *a) If $\operatorname{Re} u>0$, $\operatorname{Re} v>0$, $u\neq v$, either $z\in\mathbb{C}\backslash[1,\infty)$ and $\operatorname{Re} s>-3$, or $z=1$ and $\operatorname{Re} s>-2$, then*

$$\int_{[0,1]^3}\frac{x_1^{u-1}x_2^{v-1}x_3^{v-1}+x_1^{u-1}x_2^{u-1}x_3^{v-1}}{1-zx_1x_2x_3}(-\ln x_1x_2x_3)^s d\overline{x}=\Gamma(s+2)\frac{\Phi(z,s+2,v)-\Phi(z,s+2,u)}{u-v}.$$

b) If $\operatorname{Re} u>0$, either $z\in\mathbb{C}\backslash[1,\infty)$ and $\operatorname{Re} s>-3$, or $z=1$ and $\operatorname{Re} s>-2$, then

$$\int_{[0,1]^3}\frac{(x_1x_2x_3)^{u-1}}{1-zx_1x_2x_3}(-\ln x_1x_2x_3)^s d\overline{x}=\frac{\Gamma(s+3)}{2}\Phi(z,s+3,u).$$

c) If $\operatorname{Re} u>0$, either $z\in\mathbb{C}\backslash[1,\infty)$ and $\operatorname{Re} s>-4$, or $z=1$ and $\operatorname{Re} s>-3$, then

$$\int_{[0,1]^3}\frac{2-x_1-x_1x_2}{1-zx_1x_2x_3}(x_1x_2x_3)^{u-1}(-\ln x_1x_2x_3)^s\,d\overline{x}$$
$$=\Gamma(s+3)\left[\Phi(z,s+3,u)+\frac{(1-z)\Phi(z,s+2,u)-u^{-s-2}}{z(s+2)}\right].$$

In [2] many interesting applications of Theorems 1 and 2 are given. All of them can be generalized by Theorems 3 and 4; indeed, by these four theorems we have

$$\int_{[0,1]^2}\frac{x_1^{u-1}x_2^{v-1}}{1-zx_1x_2}(-\ln x_1x_2)^s dx_1dx_2$$
$$=(m-2)!\int_{[0,1]^m}\frac{F_{m,u,v}(x_1,x_2,\ldots,x_m)}{1-zx_1x_2\cdots x_m}(-\ln x_1x_2\cdots x_m)^{s-m+2}d\overline{x}\quad\text{for } m>1,$$

$$\int_{[0,1]^2}\frac{(x_1x_2)^{u-1}}{1-zx_1x_2}(-\ln x_1x_2)^s dx_1dx_2$$
$$=(m-1)!\int_{[0,1]^m}\frac{(x_1x_2\cdots x_m)^{u-1}}{1-zx_1x_2\cdots x_m}(-\ln x_1x_2\cdots x_m)^{s-m+2}d\overline{x}\quad\text{for } m\geq 1,$$

$$\int_{[0,1]^2}\frac{1-x_1}{1-zx_1x_2}(x_1x_2)^{u-1}(-\ln x_1x_2)^s dx_1dx_2$$
$$=(m-2)!\int_{[0,1]^m}\frac{m-1-x_1-x_1x_2-\cdots-x_1x_2\cdots x_{m-1}}{1-zx_1x_2\cdots x_m}(x_1x_2\cdots x_m)^{u-1}$$
$$\times(-\ln x_1x_2\cdots x_m)^{s-m+2}\,d\overline{x}\quad\text{for } m>1.$$

We give here only two examples.

Example 2 *Let m be an integer > 1, and $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)$ be Euler's constant. Then*

$$\gamma = (m-2)! \int_{[0,1]^m} \frac{m-1-x_1-x_1x_2-\cdots-x_1x_2\cdots x_{m-1}}{(1-x_1x_2\cdots x_m)(-\ln x_1x_2\cdots x_m)^{m-1}} d\overline{x}.$$

Example 3 *For an integer $m > 1$ the following identity holds:*

$$\ln\frac{4}{\pi} = (m-2)! \int_{[0,1]^m} \frac{m-1-x_1-x_1x_2-\cdots-x_1x_2\cdots x_{m-1}}{(1+x_1x_2\cdots x_m)(-\ln x_1x_2\cdots x_m)^{m-1}} d\overline{x}.$$

We omit details of proofs of these examples and refer to the case $m = 2$, which was considered by J. Sondow [3].

To prove Theorem 3, we will require two lemmas. The first is the identity (4) for $m = 1$, and is classical.

Lemma 1 *Suppose* $\operatorname{Re} u > 0$, *either* $z \in \mathbb{C}\backslash[1,\infty)$ *and* $\operatorname{Re} s > -1$, *or* $z = 1$ *and* $\operatorname{Re} s > 0$. *Then*

$$\int_0^1 \frac{x^{u-1}}{1-zx}(-\ln x)^s dx = \Gamma(s+1)\Phi(z, s+1, u).$$

Proof. The integral, call it I, defines a holomorphic function of z and s under the conditions stated. We can prove the statement for $|z| < 1$ and $\operatorname{Re} s > 0$ and then use analytic continuation. Expand $1/(1-zx)$ into a geometric series and then integrate:

$$I = \sum_{n=0}^{\infty} z^n \int_0^1 x^{u+n-1}(-\ln x)^s dx.$$

Making the substitution $x = e^{-y}$, we obtain

$$I = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-(u+n)y} y^s dy = \sum_{n=0}^{\infty} \frac{\Gamma(s+1)z^n}{(u+n)^{s+1}} = \Gamma(s+1)\Phi(z, s+1, u),$$

and the lemma follows.

Lemma 2 *Let $\alpha \neq 0$ and $x \in (0, 1]$. Then the following identities hold for $k \geq 1$:*
a)

$$\int_{1\geq t_1\geq t_2\geq\cdots\geq t_k\geq x} \frac{1}{t_1t_2\cdots t_k} dt_1dt_2\cdots dt_k = \frac{(-\ln x)^k}{k!}, \tag{6}$$

b)

$$\int_{1\geq t_1\geq t_2\geq\cdots\geq t_k\geq x} \frac{t_1^\alpha+t_2^\alpha+\cdots+t_k^\alpha}{t_1t_2\cdots t_k} dt_1dt_2\cdots dt_k = \frac{(-\ln x)^{k-1}}{(k-1)!}\cdot\frac{1-x^\alpha}{\alpha}. \tag{7}$$

Proof. The identity (6) is easily proved using induction and the equality

$$\int_{1\geq t_1\geq t_2\geq\cdots\geq t_k\geq x} \frac{dt_1dt_2\cdots dt_k}{t_1t_2\cdots t_k} = \int_x^1 \left(\int_{1\geq t_1\geq t_2\geq\cdots\geq t_{k-1}\geq t_k} \frac{dt_1dt_2\cdots dt_{k-1}}{t_1t_2\cdots t_{k-1}}\right)\frac{dt_k}{t_k}.$$

Denote the integral in (7) by $I_k(x)$. We prove by induction; the case $k = 1$ is true. Suppose $k > 1$ and the statement is true for $k-1$. We have

$$I_k(x) = \int_x^1 I_{k-1}(t_k)\frac{dt_k}{t_k} + \int_x^1 \left(\int_{1\geq t_1\geq t_2\geq\cdots\geq t_{k-1}\geq t_k} \frac{dt_1dt_2\cdots dt_{k-1}}{t_1t_2\cdots t_{k-1}}\right) t_k^{\alpha-1}dt_k.$$

Apply (6) to the integral in parentheses:

$$I_k(x) = \int_x^1 I_{k-1}(t_k)\frac{dt_k}{t_k} + \int_x^1 \frac{(-\ln t_k)^{k-1}}{(k-1)!} t_k^{\alpha-1} dt_k.$$

Using the induction hypothesis, we obtain

$$\begin{aligned} I_k(x) &= \int_x^1 \left(\frac{(-\ln t_k)^{k-2}}{(k-2)!} \cdot \frac{1-t_k^\alpha}{\alpha t_k} + \frac{(-\ln t_k)^{k-1}}{(k-1)!} t_k^{\alpha-1} \right) dt_k \\ &= \frac{(-\ln t_k)^{k-1}}{(k-1)!} \cdot \frac{t_k^\alpha - 1}{\alpha} \bigg|_x^1 = \frac{(-\ln x)^{k-1}}{(k-1)!} \cdot \frac{1-x^\alpha}{\alpha}. \end{aligned}$$

Now the lemma is completely proved.

Proof of Theorem 3. The integrals J_1 and J_2 in (3) and (4) define holomorphic functions of s under the conditions stated. We can prove the statement for $\operatorname{Re} s > 0$ and then use analytic continuation.

First we prove (4). Make the substitution

$$x_1 = t_1, \quad x_2 = t_2/t_1, \quad x_3 = t_3/t_2, \quad \dots, \quad x_m = t_m/t_{m-1} \tag{8}$$

in J_2. We obtain

$$\begin{aligned} J_2 &= \int_{1 \ge t_1 \ge t_2 \ge \cdots \ge t_m \ge 0} \frac{t_m^{u-1}}{1-zt_m} (-\ln t_m)^s \frac{1}{t_1 t_2 \cdots t_{m-1}} dt_1 dt_2 \cdots dt_m \\ &= \int_0^1 \frac{t_m^{u-1}}{1-zt_m} (-\ln t_m)^s \left(\int_{1 \ge t_1 \ge t_2 \ge \cdots \ge t_{m-1} \ge t_m} \frac{1}{t_1 t_2 \cdots t_{m-1}} dt_1 dt_2 \cdots dt_{m-1} \right) dt_m. \end{aligned}$$

Applying (6) with $x = t_m$ and $k = m-1$, we get

$$J_2 = \frac{1}{(m-1)!} \int_0^1 \frac{t_m^{u-1}}{1-zt_m} (-\ln t_m)^{s+m-1} dt_m.$$

It remains to apply Lemma 1.

Now we prove (3). Denote $\alpha = u - v$, then

$$J_1 = \int_{[0,1]^m} \frac{(x_1 x_2 \cdots x_m)^{v-1}}{1 - z x_1 x_2 \cdots x_m} (x_1^\alpha + (x_1 x_2)^\alpha + \cdots + (x_1 x_2 \cdots x_{m-1})^\alpha)(-\ln x_1 x_2 \cdots s_m)^s d\overline{x}.$$

Make the substitution (8)

$$J_1 = \int_0^1 \frac{t_m^{v-1}}{1-zt_m} (-\ln t_m)^s \left(\int_{1 \ge t_1 \ge t_2 \ge \cdots \ge t_{m-1} \ge t_m} \frac{t_1^\alpha + t_2^\alpha + \cdots + t_{m-1}^\alpha}{t_1 t_2 \cdots t_{m-1}} dt_1 dt_2 \cdots dt_{m-1} \right) dt_m$$

and apply (6)

$$J_1 = \frac{1}{(m-2)!\alpha} \left(\int_0^1 \frac{t_m^{v-1}}{1-zt_m} (-\ln t_m)^{s+m-2} dt_m - \int_0^1 \frac{t_m^{v+\alpha-1}}{1-zt_m} (-\ln t_m)^{s+m-2} dt_m \right).$$

It remains to apply Lemma 1 to both integrals and get back to u from α. The theorem is proved.

Remark. The formula (4) can be also obtained by letting $v \to u$ in (3) and using the identity (2).

Proof of Theorem 4. The integral J in (5) defines a function which is holomorphic in s, when $\mathrm{Re}\, s > -m-1$ if $z \in \mathbb{C}\backslash[1,\infty]$, and when $\mathrm{Re}\, s > -m$ if $z=1$. We prove the statement for $\mathrm{Re}\, s > 0$ and then use analytic continuation. We have

$$J = (m-1)\int_{[0,1]^m} \frac{(x_1x_2\cdots x_m)^{u-1}}{1-zx_1x_2\cdots x_m}(-\ln x_1x_2\cdots x_m)^s d\overline{x} \\ - \int_{[0,1]^m} \frac{F_{m,u+1,u}(x_1,x_2,\ldots,x_m)}{1-zx_1x_2\cdots x_m}(-\ln x_1x_2\cdots x_m)^s\, d\overline{x}.$$

Apply Theorem 3 to both integrals:

$$\begin{aligned} J =&(m-1)\frac{\Gamma(s+m)}{(m-1)!}\Phi(z,s+m,u) \\ &-\frac{\Gamma(s+m-1)}{(m-2)!}(\Phi(z,s+m-1,u)-\Phi(z,s+m-1,u+1)) \\ =&\frac{\Gamma(s+m)}{(m-2)!}\left[\Phi(z,s+m,u)+\frac{\Phi(z,s+m-1,u+1)-\Phi(z,s+m-1,u)}{(s+m-1)}\right]. \end{aligned}$$

Use (1) and the theorem follows.

The way which we prove Theorem 3 can be applied to any integral

$$\int_{[0,1]^m} \frac{x_1^{u_1}x_2^{u_2}\cdots x_m^{u_m}}{1-zx_1x_2\cdots x_m}(-\ln x_1x_2\cdots x_m)^s d\overline{x}.$$

We give the formula for the case when all u_i are different.

Theorem 5 *Suppose* $m \geq 1$ *and* $\mathrm{Re}\, u_1 > 0$, $\mathrm{Re}\, u_2 > 0$, $\ldots$, $\mathrm{Re}\, u_m > 0$, $u_i \neq u_j$ *whenever* $i \neq j$, *and either* $z \in \mathbb{C}\backslash[1,\infty)$ *and* $\mathrm{Re}\, s > -1$, *or* $z=1$ *and* $\mathrm{Re}\, s > 0$. *Then the following identity holds:*

$$\int_{[0,1]^m} \frac{x_1^{u_1-1}x_2^{u_2-1}\cdots x_m^{u_m-1}}{1-zx_1x_2\cdots x_m}(-\ln x_1x_2\cdots x_m)^s d\overline{x} = \Gamma(s+1)\sum_{i=1}^m \frac{\Phi(z,s+1,u_i)}{\prod_{j\neq i}(u_j-u_i)}. \tag{9}$$

To prove Theorem 5 we require the following

Lemma 3 *Let* $k \geq 1$ *and* $u_1,\ u_2,\ \ldots,\ u_{k+1}$ *be arbitrary numbers with* $u_i \neq u_j$ *whenever* $i \neq j$, *and* $x \in (0,1]$. *Then the following identity hold:*

$$\int_{1\geq t_1\geq t_2\geq\cdots\geq t_k\geq x} t_1^{u_1-u_2-1}t_2^{u_2-u_3-1}\cdots t_k^{u_k-u_{k+1}-1}dt_1dt_2\cdots dt_k = \sum_{i=1}^{k+1}\frac{x^{u_i-u_{k+1}}}{\prod_{j=1,j\neq i}^{k+1}(u_j-u_i)}. \tag{10}$$

Proof. Denote the integral in (10) by $I(u_1,u_2,\ldots,u_{k+1};x)$. We prove by induction; the case $k=1$ is true. Suppose $k>1$ and the statement is true for $k-1$, then

$$\begin{aligned} I(u_1,u_2,\ldots,u_{k+1};x) &= \int_x^1 I(u_1,u_2,\ldots,u_k;t_k)t_k^{u_k-u_{k+1}-1}dt_k \\ &= \int_x^1 \sum_{i=1}^k \frac{t_k^{u_i-u_k}}{\prod_{j=1,j\neq i}^k(u_j-u_i)}t_k^{u_k-u_{k+1}-1}dt_k \\ &= \sum_{i=1}^k \frac{1}{\prod_{j=1,j\neq i}^k(u_j-u_i)}\int_x^1 t_k^{u_i-u_{k+1}-1}dt_k \end{aligned}$$

$$= \sum_{i=1}^{k} \frac{1}{\prod_{j=1,j\neq i}^{k}(u_j - u_i)} \cdot \frac{1 - x^{u_i - u_{k+1}}}{u_i - u_{k+1}}$$

$$= \sum_{i=1}^{k} \frac{x^{u_i - u_{k+1}}}{\prod_{j=1,j\neq i}^{k+1}(u_j - u_i)} + x^{u_{k+1} - u_{k+1}} \cdot \sum_{i=1}^{k} \frac{1}{(u_i - u_{k+1}) \prod_{j=1,j\neq i}^{k}(u_j - u_i)}.$$

Thus the statement of the lemma is equivalent to the identity

$$\sum_{i=1}^{k} \frac{1}{(u_i - u_{k+1}) \prod_{j=1,j\neq i}^{k}(u_j - u_i)} = \frac{1}{\prod_{j=1}^{k}(u_j - u_{k+1})}. \tag{11}$$

To prove it, consider the polynomial

$$P(x) = \sum_{i=1}^{k} \frac{\prod_{j=1,j\neq i}^{k}(u_j - x)}{\prod_{j=1,j\neq i}^{k}(u_j - u_i)}.$$

of degree $k-1$. We have $P(u_i) = 1$ for any $i \in \{1, 2, \ldots, k\}$. Hence $P(x) \equiv 1$. The equality $P(u_{k+1}) = 1$ yields (11) and the lemma follows.

Proof of Theorem 5. In the case $m = 1$ the theorem is equivalent to Lemma 1. Now let $m > 1$. Make the substitution (8) in the integral J in (9)

$$J = \int_{1 \geq t_1 \geq t_2 \geq \cdots \geq t_m \geq 0} \frac{t_m^{u_m - 1}}{1 - z t_m} (-\ln t_m)^s t_1^{u_1 - u_2 - 1} t_2^{u_2 - u_3 - 1} \cdots t_{m-1}^{u_{m-1} - u_m - 1} dt_1 dt_2 \cdots dt_m.$$

Applying (10) for $k = m - 1$ and $x = t_m$, we obtain

$$J = \sum_{i=1}^{m} \frac{1}{\prod_{j=1,j\neq i}^{k+1}(u_j - u_i)} \int_0^1 \frac{t_m^{u_i - 1}}{1 - z t_m} (-\ln t_m)^s dt_m.$$

Use Lemma 1 and the theorem follows.

Remark. Theorem 5 is another generalization of the first equality in Theorem 1.

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References

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